Cosmological Models

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The history of the Universe is determined by the evolution of a small number of *cosmological parameters:*

$$H(t) \quad \rho_m(t) \quad \rho_r(t) \quad \rho_\Lambda = \text{cst.}$$

i.e. the evolution of the Hubble parameter, the evolution of the densities of matter, of radiation and of the density of the vacuum energy.

Moreover it is practical to introduce the following dimensionless parameters :

$$\Omega_i(t) \equiv \frac{8\pi G}{3H^2(t)}\rho_i(t)$$

That way we get the quantities $\Omega_m(t)$, $\Omega_r(t)$ and $\Omega_\Lambda(t)$ which depend on t via $H(t)$!

Introducing the critical density $\rho_{c,0}$ which corresponds to a flat Universe today, all cosmological models can be specified by the values of the density parameters which we observe today :

$$\Omega_{i,0} \equiv \frac{8\pi G}{3H_0^2}\rho_{i,0} = \frac{\rho_{i,0}}{\rho_{c,0}} \qquad \rho_{c,0} = 9.2x10^{-27}\,kgm^{-3}$$

with the cosmological parameters:

$$H_0 \quad \Omega_{m,0} \quad \Omega_{r,0} \quad \Omega_{\Lambda,0}$$

The principal goal of cosmology is to find the evolution of these parameters as a function of time. Todays' values are known with a precision of a few percent (see table at the end):

$H_0 \approx 70\ \text{kms}^{-1}\text{Mpc}^{-1}$, $\Omega_{m,0} \approx 0.3$, $\Omega_{r,0} \approx 5x10^{-5}$, $\Omega_{\Lambda,0} \approx 0.7$

The parameters of the dimensionless baryon density, the densities dark matter and neutrinos, respectively, have values which are close to:

$$\Omega_{b,0} \approx 0.05, \quad \Omega_{dm,0} \approx 0.26, \quad \Omega_{\nu,0} \approx 0,$$

With these parameters the 1st Friedmann equation can be rewritten as

$$1 = \Omega_m(t) + \Omega_r(t) + \Omega_\Lambda(t) + \Omega_k(t)$$

Where we have introduced: $\Omega_k(t) = -\frac{c^2 k}{H^2(t)a^2(t)}$

Finally we obtain Friedmann's *cosmological* equation, which is a function of todays' cosmological parameters:

$$\boxed{H^2 = H_0^2\left(\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0}a^{-2}\right)}$$

With this we can now discuss the different cosmological models, which follow from Einstein's equations and the cosmological principle.

Friedmann Model with dust (= matter) only: $\Omega_{\Lambda,0} = 0$; $\Omega_{r,0} = 0$

$$\dot{a}^2 = H_0^2\left(\Omega_{m,0}a^{-1} + 1 - \Omega_{m,0}\right) \quad \Rightarrow t = \frac{1}{H_0}\int_0^a \left[\frac{x}{\Omega_{m,0} + (1-\Omega_{m,0})x}\right]^{1/2} dx$$

For $\Omega_{m,0} = 1$ (k = 0) :

$$\boxed{a(t) = \left(\frac{3}{2}H_0 t\right)^{2/3}}$$ ....also called **Einstein-de Sitter Model**

For $\Omega_{m,0} > 1$ (k = 1) :

We set $x = \frac{\Omega_{m,0}}{(\Omega_{m,0} - 1)}\left(\sin^2 \Psi / 2\right)$ with Ψ from 0 to π

$$\boxed{a = \frac{\Omega_{m,0}}{2\left(\Omega_{m,0} - 1\right)}\left(1 - \cos\Psi\right) \qquad t = \frac{\Omega_{m,0}}{2H_0\left(\Omega_{m,0} - 1\right)^{3/2}}\left(\Psi - \sin\Psi\right)}$$

Note : a(t) evolves like a *cycloïde*

For $\Omega_{m,0} < 1$ (k < 1) :

We set $x = \frac{\Omega_{m,0}}{(\Omega_{m,0} - 1)}\left(\sinh^2 \Psi / 2\right)$

$$\boxed{a = \frac{\Omega_{m,0}}{2\left(1 - \Omega_{m,0}\right)}\left(\cosh\Psi - 1\right) \qquad t = \frac{\Omega_{m,0}}{2H_0\left(1 - \Omega_{m,0}\right)^{3/2}}\left(\sinh\Psi - \Psi\right)}$$

Friedmann Model with only radiation: $\Omega_{\Lambda,0}$ = 0; $\Omega_{m,0}$ =0

$$\dot{a}^2 = H_0^2\left(\Omega_{r,0}a^{-2} + 1 - \Omega_{r,0}\right) \quad \Rightarrow t = \frac{1}{H_0}\int_0^a \left[\frac{x}{\sqrt{\Omega_{r,0} + (1-\Omega_{r,0})x^2}}\right] dx$$

For $\Omega_{r,0}$ = 1 (k = 0) :

$$\boxed{a(t) = \left(2H_0 t\right)^{1/2}}$$

For $\Omega_{r,0}$ < 1 (k = -1) and $\Omega_{r,0}$ > 1 (k= 1) :

$$\boxed{a(t) = \left(2H_0\Omega_{r,0}^{1/2}t\right)^{1/2}\left(1 + \frac{1-\Omega_{r,0}}{2\Omega_{r,0}^{1/2}}H_0 t\right)^{1/2}}$$

Spatially flat Friedmann Models: $\Omega_{\Lambda,0}$ = 0; $\Omega_{m,0}$ +$\Omega_{r,0}$ =1; $\Omega_{k,0}$ =0;

$$\dot{a}^2 = H_0^2\left(\Omega_{m,0}a^{-1} + \Omega_{r,0}a^{-2}\right) \quad \Rightarrow t = \frac{1}{H_0}\int_0^a \left[\frac{x}{\sqrt{\Omega_{m,0}x + \Omega_{r,0}}}\right] dx$$

$$\boxed{H_0 t = \frac{2}{3\Omega_{m,0}^2}\left[\left(\Omega_{m,0}a + \Omega_{r,0}\right)^{1/2}\left(\Omega_{m,0}a - 2\Omega_{r,0}\right) + 2\Omega_{r,0}^{3/2}\right]}$$

…this cannot be easily inverted to yield *a(t)*, but it reduces to the previous results for *matter only* and *radiation only*.

The Lemaître Models = Friedmann Models + Cosmological Constant

For $\Omega_{r,0}$ = 0 and any curvature:

$$\dot{a}^2 = H_0^2\left(\Omega_{m,0}a^{-1} + \Omega_{\Lambda,0}a^2 + 1 - \Omega_{m,0} - \Omega_{\Lambda,0}\right)$$

…the general solution is complicated, but for small t one finds:

$$\boxed{a(t) = \left(\frac{3}{2}H_0\sqrt{\Omega_{m,0}}\,t\right)^{2/3}}$$

…and for large *t* the solution takes the form:

$$\boxed{a(t) \propto \exp\left(H_0 \sqrt{\Omega_{\Lambda,0}}\, t\right)}$$

Note : since there is a phase of deceleration (for small *t*) followed by a phase of acceleration (large *t*), there is an inflexion point *a** where the acceleration of *a*(*t*) vanishes :

$$a^* = \left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}\right)^{1/3}$$

Flat Lemaître Models with only matter : $\Omega_{r,0} = 0; \Omega_{m,0} + \Omega_{\Lambda,0} = 1; \Omega_{k,0} = 0;$

$$\dot{a}^2 = H_0^2\left(\left(1-\Omega_{\Lambda,0}\right)a^{-1} + \Omega_{\Lambda,0}a^2\right) \quad \Rightarrow t = \frac{1}{H_0}\int_0^a \left[\frac{x}{\sqrt{\left(1-\Omega_{\Lambda,0}\right)x + \Omega_{\Lambda,0}x^4}}\right]dx$$

This integral is a bit complicated :

We set $\quad y^2 = x^3\left|\Omega_{\Lambda,0}\right| / \left(1-\Omega_{\Lambda,0}\right)$

…which implies :

$$H_0 t = \frac{2}{3\sqrt{\left|\Omega_{\Lambda,0}\right|}} \int_0^{\sqrt{a^3\left|\Omega_{\Lambda,0}\right|/\left(1-\Omega_{\Lambda,0}\right)}} \frac{dy}{\sqrt{1 \pm y^2}}$$

Where the + (-) signs correspond to $\Omega_{\Lambda,0} > 0$ and (< 0), respectively. After doing the integrals we find:

$$H_0 t = \frac{2}{3\sqrt{\left|\Omega_{\Lambda,0}\right|}} \begin{Bmatrix} \sinh^{-1}\left[\sqrt{a^3\left|\Omega_{\Lambda,0}\right|/\left(1-\Omega_{\Lambda,0}\right)}\right] & \Omega_{\Lambda,0} > 0 \\ \sin^{-1}\left[\sqrt{a^3\left|\Omega_{\Lambda,0}\right|/\left(1-\Omega_{\Lambda,0}\right)}\right] & \Omega_{\Lambda,0} < 0 \end{Bmatrix}$$

The de Sitter Model = Lemaître Model with $\Omega_{r,0}$ = 0; $\Omega_{m,0}$ = 0; $\Omega_{\Lambda,0}$ = 1; k=0

$$\dot{a}^2 = H_0^2 a^2$$

$\rightarrow$ The Hubble parameter is now a true constant and the scaling parameter grows exponentially:

$$a(t) = \exp\left[H_0(t-t_0)\right] = \exp\left[\sqrt{\Lambda/3}c(t-t_0)\right]$$

$\rightarrow$ The de Sitter model does not have a singularity of the Big Bang type at a finite time and distance in the past.

Einstein's static universe

This model is interesting for historical reasons! Einstein, looking for a static universe, was setting $\dot{a} = \ddot{a} = 0$ and introduced a cosmological constant with $\Lambda > 0$. Doing so, the Hubble parameter is always zero and the other parameters are infinite. For Λ one gets the relation :

$$4\pi G\rho_{m,0} = \Lambda c^2 = \frac{c^2 k}{a_0^2}$$

or $\rho_{m,0} = \Lambda c^2/(4\pi G) = 2\rho_{\Lambda,0}$. Since $\Lambda > 0$, we have also k = +1. With $\rho_{m,0} \approx 3\text{x}10^{-27}$ kg/m^3 one finds $a_0 \approx 2\text{x } 10^{26}$ m $\approx$ 6000 Mpc and $\Lambda \approx 2.5 \text{ x } 10^{-53}$ m^{-2}. These values could not be verified experimentally at that epoch. However the model requires an extremely fine adjustment of Λ in order to obtain a static Universe, in fact one obtains an unstable equilibrium.

Einstein : « my biggest blunder»

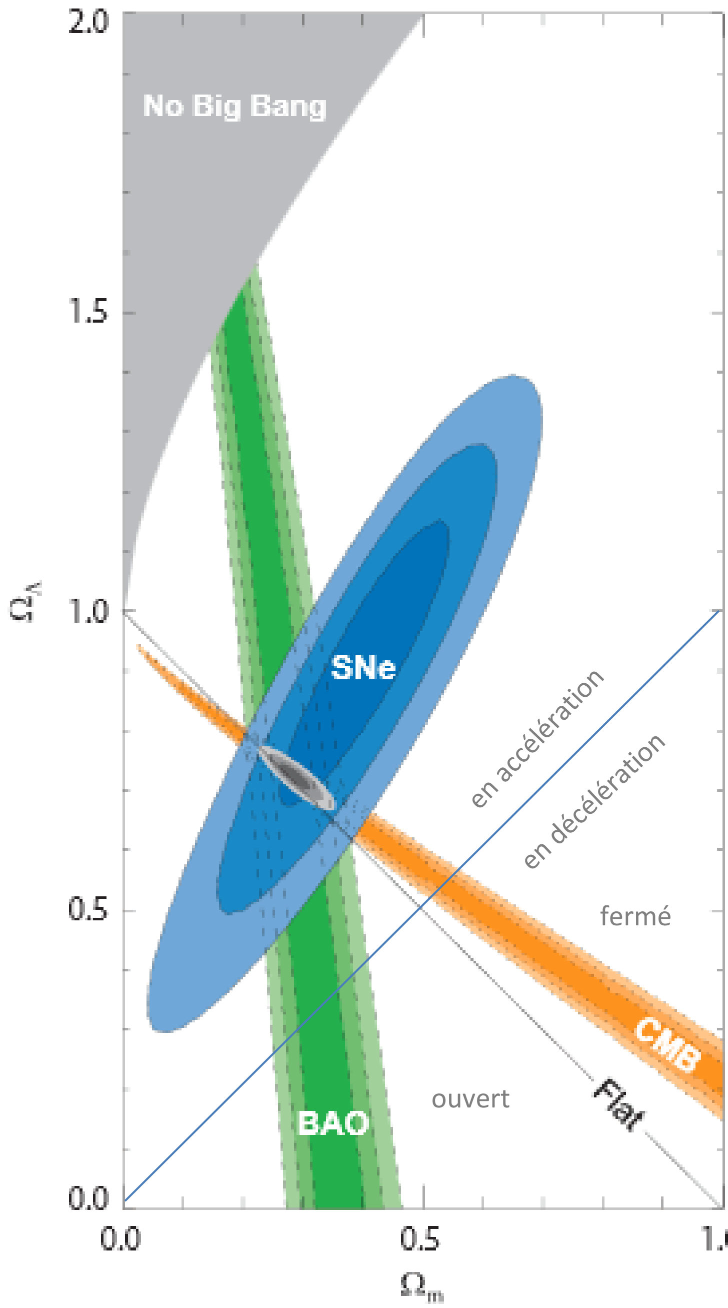

Figure 23.1: Confidence level contours of 68.3%, 95.4% and 99.7% in the Ω_Λ–Ω_m plane from the CMB, BAOs and the Union SNe Ia set, as well as their combination (assuming $w = -1$). [Courtesy of Kowalski *et al.* [25]]

(Ref. : http://pdg.web.cern.ch/pdg/2012/reviews/rpp2012-rev-cosmological-parameters.pdf)

Table 23.1: The basic set of cosmological parameters. We give values (with some additional rounding) as obtained using a fit of a spatially-flat ΛCDM cosmology with a power-law initial spectrum to WMAP7 data alone: Table 10, left column of Ref. 2. Tensors are assumed zero except in quoting a limit on them. The exact values and uncertainties depend on both the precise data-sets used and the choice of parameters allowed to vary. Limits on Ω_Λ and h weaken if the Universe is not assumed flat. The density perturbation amplitude is specified by the derived parameter σ_8. Uncertainties are one-sigma/68% confidence unless otherwise stated.

Parameter	Symbol	Value
Hubble parameter	h	0.704 ± 0.025
Cold dark matter density	$\Omega_{\rm cdm}$	$\Omega_{\rm cdm}h^2 = 0.112 \pm 0.006$
Baryon density	$\Omega_{\rm b}$	$\Omega_{\rm b}h^2 = 0.0225 \pm 0.0006$
Cosmological constant	Ω_Λ	0.73 ± 0.03
Radiation density	$\Omega_{\rm r}$	$\Omega_{\rm r}h^2 = 2.47 \times 10^{-5}$
Neutrino density	Ω_ν	See Sec. 23.1.2
Density perturb. amplitude at $k = 0.002\mathrm{Mpc}^{-1}$	$\Delta^2_{\mathcal{R}}$	$(2.43 \pm 0.11) \times 10^{-9}$
Density perturb. spectral index	n	0.967 ± 0.014
Tensor to scalar ratio	r	$r < 0.36$ (95% conf.)
Ionization optical depth	τ	0.088 ± 0.015
Bias parameter	b	See Sec. 23.3.4

(Ref. : http://pdg.web.cern.ch/pdg/2012/reviews/rpp2012-rev-cosmological-parameters.pdf)